

A unified framework for discrete cohomology and bounded interpolation

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Hilbert complex and cohomology

Hilbert complex:

$$0 \longrightarrow W^0 \xrightarrow{d^0} W^1 \xrightarrow{d^1} \dots \xrightarrow{d^{m-1}} W^m \longrightarrow 0$$

- W^k : Hilbert space.
- $d^k W^k \subset W^{k+1}$ is a closed subspace, and $d^k \circ d^{k+1} = 0$.
- **Cohomology:**

$$\mathcal{H}^k := \frac{\ker(d^k : W^k \rightarrow W^{k+1})}{\operatorname{im}(d^{k-1} : W^{k-1} \rightarrow W^k)}.$$

- **Exactness:** if $\ker(d^k : W^k \rightarrow W^{k+1}) = \operatorname{im}(d^{k-1} : W^{k-1} \rightarrow W^k)$.

de Rham complex in $\mathbb{R}^3$:

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\operatorname{grad}} H(\operatorname{curl}, \Omega) \xrightarrow{\operatorname{curl}} H(\operatorname{div}, \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega) \longrightarrow 0$$

Betti numbers:

$$\beta_k = \dim \mathcal{H}_{dR}^k(\Omega),$$

which characterize the topology of Ω .

Finite element de Rham complex

Betti numbers for some domains in $\mathbb{R}^3$:

β_0 : connected components β_1 : tunnels β_2 : cavities		
1 Solid ball  $\beta_0 = 1, \beta_1 = 0, \beta_2 = 0$	2 Two disjoint solid balls  $\beta_0 = 2, \beta_1 = 0, \beta_2 = 0$	3 Solid torus  $\beta_0 = 1, \beta_1 = 1, \beta_2 = 0$
4 Genus-2 handlebody  $\beta_0 = 1, \beta_1 = 2, \beta_2 = 0$	5 Spherical shell  $\beta_0 = 1, \beta_1 = 0, \beta_2 = 1$	6 Region with two cavities  $\beta_0 = 1, \beta_1 = 0, \beta_2 = 2$

Discrete de Rham complex:

$$0 \longrightarrow \mathcal{L}_1 \xrightarrow{\text{grad}} \mathcal{N}_0 \xrightarrow{\text{curl}} \mathcal{RT}_0 \xrightarrow{\text{div}} \mathcal{P}_T^0 \longrightarrow 0$$

Cohomology preservation:

$$\mathcal{H}_{dR,h}^k(\Omega) \cong \mathcal{H}_{dR}^k(\Omega), \quad \beta_{k,h} = \beta_k.$$

Refs: [Arnold & Falk & Winther, 2006], [Christiansen & Winther, 2008], [Licht, 2017].

Commuting interpolation

Commuting interpolation

$$\begin{array}{ccccccc} 0 & \longrightarrow & W^0 & \xrightarrow{d^0} & W^1 & \xrightarrow{d^1} & \cdots \xrightarrow{d^{n-1}} W^n \longrightarrow 0 \\ & & \pi^0 \downarrow & & \pi^1 \downarrow & & \downarrow & & \pi^n \downarrow \\ 0 & \longrightarrow & W_h^0 & \xrightarrow{d^0} & W_h^1 & \xrightarrow{d^1} & \cdots \xrightarrow{d^{n-1}} W_h^n \longrightarrow 0 \end{array}$$

Property of π^k

- **Bounded:** $\|\pi^k u\|_{H(d^k)} \leq C \|u\|_{H(d^k)}$ or $\|\pi^k u\|_{L^2} \leq C \|u\|_{L^2}$.
- **Commuting:** $\pi^{k+1} \circ d^k = d^k \circ \pi^k$.
- **Projection:** $\pi^k u_h = u_h$ for any $u_h \in W_h^k$.
- **Local:** For each element T , $\pi^k u|_T$ depend solely on a patch of T

Exsting interploation

- Non-local: [Schöberl, 2005], [Arnold & Falk & Winther, 2006], [Christiansen & Winther, 2008], etc.
- Local: [Falk & Winther, 2014, 2015], [Arnold & Guzmán, 2021], [Ern & Guzmán & Potu & Vohralík, 2025]

Other complexes

BGG complexes ([Arnold & Hu, 2021])

- **Hessian complex**

$$0 \longrightarrow H^2(\Omega) \xrightarrow{\text{hess}} H(\text{curl}, \Omega; \mathbb{S}) \xrightarrow{\text{curl}} H(\text{div}, \Omega; \mathbb{T}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^3) \longrightarrow 0$$

- **divdiv complex**

$$0 \longrightarrow H^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{dev grad}} H(\text{sym curl}, \Omega; \mathbb{T}) \xrightarrow{\text{sym curl}} H(\text{div div}, \Omega; \mathbb{S}) \xrightarrow{\text{div div}} L^2(\Omega) \longrightarrow 0$$

- **elasticity complex**

$$0 \longrightarrow H^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{sym grad}} H(\text{inc}, \Omega; \mathbb{S}) \xrightarrow{\text{inc}} H(\text{div}, \Omega; \mathbb{S}) \xrightarrow{\text{div}} L^2(\Omega; \mathbb{R}^3) \longrightarrow 0$$

- **conformal deformation complex, conformal Hessian complex**

Extra smoothness

- **Stokes complex**

$$0 \longrightarrow H^2(\Omega) \xrightarrow{\text{grad}} H^1(\text{curl}, \Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} H^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0$$

Finite element complexes

Finite element discretizations

- Hessian complex: [Hu & Liang, 2021], etc.
- divdiv complex: [Chen & Huang, 2022divdiv], [Hu & Liang & Ma, 2022], [Hu & Liang & Ma & Zhang, 2024], etc.
- elasticity complex: [Christiansen & Hu & Hu, 2018], [Chen & Huang, 2022elasticity], [Christiansen & Gopalakrishnan & Guzmán & Hu, 2024], [Guzmán & Huang, 2026], etc.
- Stokes complex: [Falk & Neilan, 2013], [Neilan, 2015], [Fu & Guzmán & Neilan, 2020], [Hu & Zhang & Zhang, 2022], [Guzmán & Lischke & Neilan, 2022], etc.
- conformal deformation complex: [Hu & Lin & Shi, 2023], [Huang, 2025]
- conformal Hessian complex: [Guo & Hu & Lin, 2025], [Huang, 2025]

Exactness: These finite element complexes are exact on topologically trivial domains.

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Question I: What is the finite element cohomology on domains with nontrivial topology?

Finite element complexes

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- divdiv complex: [Chen & Huang, 2022divdiv], [Hu & Liang & Ma, 2022], [Hu & Liang & Ma & Zhang, 2024], etc.
- elasticity complex: [Christiansen & Hu & Hu, 2018], [Chen & Huang, 2022elasticity], [Christiansen & Gopalakrishnan & Guzmán & Hu, 2024], [Guzmán & Huang, 2026], etc.
- Stokes complex: [Falk & Neilan, 2013], [Neilan, 2015], [Fu & Guzmán & Neilan, 2020], [Hu & Zhang & Zhang, 2022], [Guzmán & Lischke & Neilan, 2022], etc.
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Exactness: These finite element complexes are exact on topologically trivial domains.

Question I: What is the finite element cohomology on domains with nontrivial topology?

Question II: How to construct local bounded commuting projections for nonstandard finite element complexes?

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Key ingredient I: Generalized de Rham map

For each smooth BGG complex:

$$0 \longrightarrow C^\infty(\Omega; \mathbb{X}^0) \xrightarrow{d^0} C^\infty(\Omega; \mathbb{X}^1) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} C^\infty(\Omega; \mathbb{X}^n) \longrightarrow 0$$

Key ingredient I: Generalized de Rham map (generalized currents)

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^\infty(\Omega; \mathbb{X}^0) & \xrightarrow{d^0} & C^\infty(\Omega; \mathbb{X}^1) & \xrightarrow{d^1} & \dots \xrightarrow{d^{n-1}} C^\infty(\Omega; \mathbb{X}^n) \longrightarrow 0 \\ & & \Upsilon_0 \downarrow & & \Upsilon_1 \downarrow & & \cdot \downarrow & & \Upsilon_n \downarrow \\ 0 & \longrightarrow & \mathcal{Z} \otimes \Delta'_0 & \xrightarrow{\delta} & \mathcal{Z} \otimes \Delta'_1 & \xrightarrow{\delta} & \dots \xrightarrow{\delta} & \mathcal{Z} \otimes \Delta'_n \longrightarrow 0 \end{array}$$

Properties of Υ_i :

- (1) **Surjectivity:** $\Upsilon_i : C^\infty(\Omega; \mathbb{X}^i) \rightarrow \mathcal{Z} \otimes \Delta'_i$.
- (2) **Commuting property:** $\delta \circ \Upsilon_i = \Upsilon_{i+1} \circ d^i$.

Key ingredient II: Geometric decomposition

Key ingredient II: Geometric decomposition of finite element complex

Finite element complex

$$0 \longrightarrow \mathbf{A}^0 \xrightarrow{d^0} \mathbf{A}^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \mathbf{A}^n \longrightarrow 0$$

is a **direct sum** of the **Skeletal complex**

$$0 \longrightarrow \mathbf{S}^0(\mathcal{T}) \xrightarrow{d^0} \mathbf{S}^1(\mathcal{T}) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \mathbf{S}^n(\mathcal{T}) \longrightarrow 0$$

and the **Bubble complex**

$$0 \longrightarrow \mathbb{B}^0(\tau) \xrightarrow{d^0} \mathbb{B}^1(\tau) \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \mathbb{B}^n(\tau) \longrightarrow 0$$

with all $\tau \in \Delta(\mathcal{T})$.

Cohomology of finite element complex

Cohomology of subcomplex

- Skeletal complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & S^0(\mathcal{T}) & \xrightarrow{d^0} & S^1(\mathcal{T}) & \xrightarrow{d^1} & \cdots \xrightarrow{d^{n-1}} S^n(\mathcal{T}) \longrightarrow 0 \\ & & \downarrow \Upsilon_0 & & \downarrow \Upsilon_1 & & \downarrow \cdot & & \downarrow \Upsilon_3 \\ 0 & \longrightarrow & \mathcal{Z} \otimes \Delta'_0 & \xrightarrow{\delta} & \mathcal{Z} \otimes \Delta'_1 & \xrightarrow{\delta} & \cdots \xrightarrow{\delta} \mathcal{Z} \otimes \Delta'_n \longrightarrow 0 \end{array}$$

here $S^k(\mathcal{T}) \stackrel{\Upsilon_k}{\cong} \mathcal{Z} \otimes \Delta'_k, 0 \leq k \leq n$.

- Bubble complex

$$0 \longrightarrow \mathbb{B}^0(\tau) \xrightarrow{d^0} \mathbb{B}^1(\tau) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathbb{B}^n(\tau) \longrightarrow 0$$

is local and exact.

Conclusion I: The cohomology of the finite element complex

$$0 \longrightarrow \mathbf{A}^0 \xrightarrow{d^0} \mathbf{A}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathbf{A}^n \longrightarrow 0$$

is isomorphic to $\mathcal{Z} \otimes \mathcal{H}_{dR}(\Omega)$.

Commuting interpolations for finite element complex

Conclusion II: Construction of local bounded commuting projections $\pi^k : L^2(\Omega; \mathbb{X}^k) \rightarrow \mathbf{A}^k$:

$$\pi^k(w) = \pi_{\mathcal{S}}^k(w) + \sum_{\tau \in \Delta(\mathcal{T})} \pi_{\mathbf{B},\tau}^k(w),$$

here

$$\pi_{\mathcal{S}}^k(w) = \sum_{\tau \in \Delta_k} \sum_{i=1}^{\dim \mathcal{Z}} (w, \Xi_{\tau}^i)_{\text{star}^1(\tau)} \mathbf{E}_{\mathcal{S}}^k(z_i \otimes \tau) \in S^k(\mathcal{T}),$$

and

$$\begin{aligned} \pi_{\mathbf{B},\tau}^k(w) = & \sum_{i=1}^{\mathcal{N}_{\tau,k}} (w, d^* \Omega_{\tau,k+1}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(\phi_{\tau,k}^i) + \\ & \sum_{i=1}^{\mathcal{N}_{\tau,k-1}} (w, \Omega_{\tau,k}^i)_{\text{star}(\tau)} \mathbf{E}_{\tau}^k(d\phi_{\tau,k-1}^i) \in \mathbb{B}^k(\tau). \end{aligned}$$

with some $\Xi_{\tau}^i \in H_0^M(\text{star}^1(\tau)) \otimes \mathbb{X}^{\dim \tau}$ and $\Omega_{\tau,k}^i \in C_0^{2M}(\text{star}(\tau)) \otimes \mathbb{X}^k$.

Ref: [Arnold & Guzmán, 2021], [Ern & Guzmán & Potu & Vohralík, 2025].

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3D Hessian complex

Consider the following finite element 3D Hessian complex ([Hu & Liang, 2021])

$$0 \longrightarrow U_h \xrightarrow{\text{hess}} \Sigma_h \xrightarrow{\text{curl}} V_h \xrightarrow{\text{div}} Q_h \longrightarrow 0$$

For $k \geq 9$, here

$$U_h = \{u \in H^2(\Omega) : u|_T \in P_k(T), u \text{ is } C^4 \text{ at each vertex } x \in \Delta_0, \\ C^2 \text{ on each edge } e \in \Delta_1\}$$

$$\Sigma_h = \{\boldsymbol{\sigma} \in H(\text{curl}, \Omega; \mathbb{S}) : \boldsymbol{\sigma}|_T \in P_{k-2}(T; \mathbb{S}), \boldsymbol{\sigma} \text{ is } C^2 \text{ at each vertex } x \in \Delta_0, \\ C^0 \text{ on each edge } e \in \Delta_1\}$$

$$V_h = \{\mathbf{v} \in H(\text{div}, \Omega; \mathbb{T}) : \mathbf{v}|_T \in P_{k-3}(T; \mathbb{T}), \boldsymbol{\sigma} \text{ is } C^1 \text{ at each vertex } x \in \Delta_0\}$$

$$Q_h = \{\mathbf{q} \in L^2(\Omega; \mathbb{R}^3) : \mathbf{q}|_T \in P_{k-4}(T; \mathbb{R}^3), \boldsymbol{\sigma} \text{ is } C^0 \text{ at each vertex } x \in \Delta_0\}$$

Another degrees of freedom

	U_h	$\xrightarrow{\text{hess}}$	Σ_h	$\xrightarrow{\text{curl}}$	V_h	$\xrightarrow{\text{div}}$	Q_h
Skeletal	$\Upsilon_x(u)$		$\Upsilon_e(\sigma)$		$\Upsilon_f(v)$		$\Upsilon_T(q)$
Vertex	$D^\alpha u(x), 2 \leq \alpha \leq 4$		$D^\beta \sigma(x), \beta \leq 2$		$v(x), \nabla v(x)$		$q(x)$
Edge(1)	$(\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 b_e}{\partial t^2})_e$		$(t^T \sigma t, b_e^2)_e$				
Edge(2)	$(\frac{\partial^2 u}{\partial n \partial t}, \frac{\partial b_e^{1,*}}{\partial t})_e$		$(n_\pm^T \sigma t, b_e^{2,*})_e$				
Edge(3)	$(\frac{\partial^2 u}{\partial n_\pm \partial n_\pm}, \frac{\partial b_e^{1,**}}{\partial t})_e$		$(n_\pm^T \sigma n_\pm, b_e^{2,**})_e$				
Face(1)	$(\nabla_f^2 u, \nabla_f^2 b_f^1)_f$		$\langle \sigma, b_f^2 \rangle_f^1$		$(E_f(vn), b_f^3)_f$		
Face(2)	$(\nabla_f(\frac{\partial u}{\partial n}), \nabla_f b_f^{1,*})_f$		$\langle \sigma, b_f^{2,*} \rangle_f^2$		$(n^T v n, b_f^{3,*})_f$		
Tetrahedron	$(\nabla^2 u, \nabla^2 b_T^1)_T$		$\langle \sigma, b_T^2 \rangle_T^1$		$\langle v, b_T^3 \rangle_T^2$		$(q, b_T^4)_T$

Generalized de Rham maps: let $\{\psi_i\}_{1 \leq i \leq 4}$ be a basis of $\mathcal{RT} = \{a + bx : a \in \mathbb{R}^3, b \in \mathbb{R}\}$

$$\Upsilon_x : U_h \rightarrow \mathbb{R}^4, \quad [\Upsilon_x(u)]_i = (\psi_i \cdot \nabla u)(x) - \frac{1}{3}(\text{div } \psi_i \cdot u)(x),$$

$$\Upsilon_e : \Sigma_h \rightarrow \mathbb{R}^4, \quad [\Upsilon_e(\sigma)]_i = \int_e (\sigma t) \cdot \psi_i,$$

$$\Upsilon_f : V_h \rightarrow \mathbb{R}^4, \quad [\Upsilon_f(\tau)]_i = \int_f (\tau n) \cdot \psi_i,$$

$$\Upsilon_T : Q_h \rightarrow \mathbb{R}^4, \quad [\Upsilon_T(q)]_i = \int_T q \cdot \psi_i.$$

Separation of DOFs

- For each $u \in U_h$, $\text{hess } u \in \Sigma_h$ and

“**Skeletal**” DOFs of u vanish $\Rightarrow$ “**Skeletal**” DOFs of $\text{hess } u$ vanish

- For each vertex $x \in \Delta_0$

“**Vertex**” DOFs of u at x vanish $\Rightarrow$ “**Vertex**” DOFs of $\text{hess } u$ at x vanish

- For each edge $e \in \Delta_1$

“**Edge**” DOFs of u on e vanish $\Rightarrow$ “**Edge**” DOFs of $\text{hess } u$ on e vanish

- For each face $f \in \Delta_2$

“**Face**” DOFs of u on f vanish $\Rightarrow$ “**Face**” DOFs of $\text{hess } u$ on f vanish

- For each face $T \in \Delta_3$

“**Tetrahedron**” DOFs of u on T vanish $\Rightarrow$ “**Tetrahedron**” DOFs of $\text{hess } u$
on T vanish

This also holds for each $\sigma \in \Sigma_h$ and $v \in V_h$.

Geometric decomposition of complex

Geometric decomposition of finite elements

$$U_h := U^S \oplus (\oplus_{x \in \Delta_0} U^x) \oplus (\oplus_{e \in \Delta_1} U^e) \oplus (\oplus_{f \in \Delta_2} U^f) \oplus (\oplus_{T \in \Delta_3} U^T)$$

$$\Sigma_h := \Sigma^S \oplus (\oplus_{x \in \Delta_0} \Sigma^x) \oplus (\oplus_{e \in \Delta_1} \Sigma^e) \oplus (\oplus_{f \in \Delta_2} \Sigma^f) \oplus (\oplus_{T \in \Delta_3} \Sigma^T)$$

$$V_h := V^S \oplus (\oplus_{x \in \Delta_0} V^x) \oplus (\oplus_{e \in \Delta_1} V^e) \oplus (\oplus_{f \in \Delta_2} V^f) \oplus (\oplus_{T \in \Delta_3} V^T)$$

$$Q_h := Q^S \oplus (\oplus_{x \in \Delta_0} Q^x) \oplus (\oplus_{e \in \Delta_1} Q^e) \oplus (\oplus_{f \in \Delta_2} Q^f) \oplus (\oplus_{T \in \Delta_3} Q^T)$$

Geometric decomposition of 3D Hessian complex: The 3D Hessian complex

$$0 \longrightarrow U_h \xrightarrow{\text{hess}} \Sigma_h \xrightarrow{\text{curl}} V_h \xrightarrow{\text{div}} Q_h \longrightarrow 0$$

is a **direct sum** of the **Skeletal complex**

$$0 \longrightarrow U^S \xrightarrow{\text{hess}} \Sigma^S \xrightarrow{\text{curl}} V^S \xrightarrow{\text{div}} Q^S \longrightarrow 0$$

and **Bubble complexes**

$$0 \longrightarrow U^\tau \xrightarrow{\text{hess}} \Sigma^\tau \xrightarrow{\text{curl}} V^\tau \xrightarrow{\text{div}} Q^\tau \longrightarrow 0$$

with each $\tau \in \Delta(\mathcal{T}_h)$.

Cohomology of 3D Hessian complex

- For each $\tau \in \Delta(\mathcal{T}_h)$, the [Bubble complex](#)

$$0 \longrightarrow U^\tau \xrightarrow{\text{hess}} \Sigma^\tau \xrightarrow{\text{curl}} V^\tau \xrightarrow{\text{div}} Q^\tau \longrightarrow 0$$

is **local and exact**.

- The [Skeletal complex](#)

$$0 \longrightarrow U^S \xrightarrow{\text{hess}} \Sigma^S \xrightarrow{\text{curl}} V^S \xrightarrow{\text{div}} Q^S \longrightarrow 0$$

is **isomorphic to the simplicial cochain complex**

$$0 \longrightarrow \mathbb{R}^4 \otimes \Delta'_0 \xrightarrow{\delta} \mathbb{R}^4 \otimes \Delta'_1 \xrightarrow{\delta} \mathbb{R}^4 \otimes \Delta'_2 \xrightarrow{\delta} \mathbb{R}^4 \otimes \Delta'_3 \longrightarrow 0$$

the isomorphism is deduced by $\{\Upsilon_\bullet\}$.

- Conclusion:** The cohomology of the 3D finite element Hessian complex is isomorphic to $\mathbb{R}^4 \otimes \mathcal{H}_{dR}(\Omega)$.

3D Stokes complex

Consider the following 3D with mixed boundary conditions

$$0 \longrightarrow H_D^2(\Omega) \xrightarrow{\text{grad}} H_D^1(\Omega; \text{curl}) \xrightarrow{\text{curl}} (H_D^1(\Omega))^3 \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0 \quad (1)$$

with

$$\begin{aligned} H_D^k(\Omega) &:= \{\phi \in H^k(\Omega) : D^\alpha \phi|_{\Gamma_D} = 0, \forall |\alpha| \leq k-1\}, \\ H_D^1(\Omega; \text{curl}) &:= \{v \in H_D^1(\Omega)^3 : \text{curl } v \in H_D^1(\Omega)^3\}, \end{aligned}$$

Cohomology:

The cohomology of (1) is isomorphic to $\mathcal{H}_{dR,D}(\Omega)$

Finite element Stokes complex on Alfeld splits

The finite element Stokes complex on Alfeld splits ([Fu & Guzmán & Neilan, 2020])

$$0 \longrightarrow M_{d,r}^0(\mathcal{T}_h^z) \xrightarrow{\text{grad}} M_{d,r-1}^1(\mathcal{T}_h^z) \xrightarrow{\text{curl}} M_{r-2}^2(\mathcal{T}_h^z) \xrightarrow{\text{div}} V_{d,r-3}^3(\mathcal{T}_h^z) \longrightarrow 0.$$

For $r \geq 5$, here

$$M_{d,r}^0(\mathcal{T}_h^z) = \{u \in H_D^2(\Omega) : u|_K \in P_r(K_A), K \in \mathcal{T}_h, u \text{ is } C^2 \text{ at each vertex } x \in \Delta_0\}$$

$$M_{d,r-1}^1(\mathcal{T}_h^z) = \{\mathbf{v} \in H_D^1(\Omega; \text{curl}) : \mathbf{v}|_K \in P_{r-1}(K_A; \mathbb{R}^3), K \in \mathcal{T}_h, \mathbf{v} \text{ is } C^1 \text{ at each vertex } x \in \Delta_0\}$$

$$M_{r-2}^2(\mathcal{T}_h^z) = \{\mathbf{v} \in (H_D^1(\Omega))^3 : \mathbf{v}|_K \in P_{r-2}(K_A; \mathbb{R}^3), K \in \mathcal{T}_h\}$$

$$V_{d,r-3}^3(\mathcal{T}_h^z) = \{q \in L^2(\Omega) : q|_K \in P_{r-3}(K_A), K \in \mathcal{T}_h\}.$$

Theorem 1 (Brubeck & Liang & Parker, 2026)

The *cohomology* of the finite element Stokes complex on Alfeld splits ([Fu, Guzmán, Neilan, 2020]) with mixed boundary conditions is *isomorphic to* $\mathcal{H}_{dR,D}(\Omega)$.

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Conclusion and Extension

Conclusion: We provide a unified framework to investigate the **cohomology and bounded interpolation** of nonstandard finite element complexes, e.g. Stokes, Hessian, elasticity, divdiv.

Possible extension:

- Boundary treatment. ([Brubeck & Liang & Parker, 2026])
- p -version, spline function, distributional function.
- Other BGG complexes whose cohomology is not isomorphic to $\mathcal{Z} \otimes \mathcal{H}_{dR}(\Omega)$, e.g., the curl-div and grad-curl complex.
- Other meshes.

Reference:

- J. Hu, Y. Liang, and T. Lin, Finite Element Complexes with Traces Structures: A unified framework for cohomology and bounded interpolation.[arXiv:2509.23788]

Thanks!