Finite Element Complexes with Traces Structures: A unified framework for cohomology and bounded interpolation

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Theory of Differential Complexes and Related Models, ESI

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Cohomolgy of finite element de Rham complex

- **de Rham complex**

$$0 \longrightarrow \Lambda^0(\Omega) \xrightarrow{d^0} \Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Lambda^n(\Omega) \longrightarrow 0$$

  de Rham cohomology $\mathcal{H}^k_{dR}(\Omega) := \frac{\ker(d:\Lambda^k(\Omega)\to\Lambda^{k+1}(\Omega))}{\operatorname{im}(d:\Lambda^{k-1}(\Omega)\to\Lambda^k(\Omega))}$.

- **Finite element differential forms** on simplicial meshes:

$$0 \longrightarrow \Lambda^0_h(\Omega) \xrightarrow{d^0} \Lambda^1_h(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Lambda^n_h(\Omega) \longrightarrow 0$$

  The discrete de Rham cohomology $\mathcal{H}^k_{dR,h}(\Omega) := \frac{\ker(d:\Lambda^k_h(\Omega)\to\Lambda^{k+1}_h(\Omega))}{\operatorname{im}(d:\Lambda^{k-1}_h(\Omega)\to\Lambda^k_h(\Omega))}$ is isomorphic to $\mathcal{H}^k_{dR}(\Omega)$.

- **Ref**: [Arnold & Falk & Winther, 2006], [Christiansen & Winther, 2008], [Licht, 2017], etc.

de Rham map

Consider the case of $\Omega \subset \mathbb{R}^3$, let $\Delta(\mathcal{T}_h)$ be a triangulation (simplicial complex) of Ω, the set of all oriented k-simplicies of $\Delta(\mathcal{T}_h)$ is denoted by Δ_k.

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(\Omega) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Upsilon_0"'] & C^\infty(\Omega;\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Upsilon_1"'] & C^\infty(\Omega;\mathbb{R}^3) \arrow[r, "\mathrm{div}"] \arrow[d, "\Upsilon_2"'] & C^\infty(\Omega) \arrow[r] \arrow[d, "\Upsilon_3"'] & 0 \\
0 \arrow[r] & \mathbb{R}\otimes\Delta_0' \arrow[r, "\delta"] & \mathbb{R}\otimes\Delta_1' \arrow[r, "\delta"] & \mathbb{R}\otimes\Delta_2' \arrow[r, "\delta"] & \mathbb{R}\otimes\Delta_3' \arrow[r] & 0
\end{tikzcd}$$

$$\Upsilon_0[\phi](x) = \phi(x), \quad x \in \Delta_0, \qquad \Upsilon_1[\boldsymbol{\phi}](e) = \int_e \boldsymbol{\phi}\cdot t_e, \quad e \in \Delta_1$$

$$\Upsilon_2[\boldsymbol{\phi}](f) = \int_f \boldsymbol{\phi}\cdot n_f, \quad f \in \Delta_2, \qquad \Upsilon_3[\phi](T) = \int_T \phi, \quad T \in \Delta_3$$

There holds that

$$\delta\circ\Upsilon_0 = \Upsilon_1\circ\mathrm{grad}, \quad \delta\circ\Upsilon_1 = \Upsilon_2\circ\mathrm{curl}, \quad \delta\circ\Upsilon_2 = \Upsilon_3\circ\mathrm{div}$$

From de Rham's theorem: *The de Rham cohomology spaces are isomorphic to the simplicial cohomology spaces. Moreover, the isomorphism is induced by* $\{\Upsilon_\bullet\}$.

The Whitney forms

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}_-^1\Lambda^0(\mathcal{T}_h) \arrow[r, "\mathrm{grad}"] \arrow[d, "\Upsilon_0"'] & \mathcal{P}_-^1\Lambda^1(\mathcal{T}_h) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Upsilon_1"'] & \mathcal{P}_-^1\Lambda^2(\mathcal{T}_h) \arrow[r, "\mathrm{div}"] \arrow[d, "\Upsilon_2"'] & \mathcal{P}_-^1\Lambda^3(\mathcal{T}_h) \arrow[r] \arrow[d, "\Upsilon_3"'] & 0 \\
0 \arrow[r] & \mathbb{R}\otimes\Delta_0' \arrow[r, "\delta"] & \mathbb{R}\otimes\Delta_1' \arrow[r, "\delta"] & \mathbb{R}\otimes\Delta_2' \arrow[r, "\delta"] & \mathbb{R}\otimes\Delta_3' \arrow[r] & 0
\end{tikzcd}$$

Note that Υ_i commutes with differential operators and

$$\mathcal{P}_-^1\Lambda^i(\mathcal{T}_h) \overset{\Upsilon_i}{\cong} \mathbb{R}\otimes\Delta_i', \quad 0 \leq i \leq 3$$

Result: The cohomology spaces of the Whitney forms are isomorphic to the de Rham cohomology spaces.

Commuting interpolation

Commuting interpolation

$$\begin{tikzcd}
0 \arrow[r] & L^2\Lambda^0(\Omega) \arrow[r, "d^0"] \arrow[d, "\pi^0"'] & L^2\Lambda^1(\Omega) \arrow[r, "d^1"] \arrow[d, "\pi^1"'] & \cdots \arrow[r, "d^{n-1}"] \arrow[d] & L^2\Lambda^n(\Omega) \arrow[r] \arrow[d, "\pi^n"'] & 0 \\
0 \arrow[r] & \Lambda_h^0(\Omega) \arrow[r, "d^0"] & \Lambda_h^1(\Omega) \arrow[r, "d^1"] & \cdots \arrow[r, "d^{n-1}"] & \Lambda_h^n(\Omega) \arrow[r] & 0
\end{tikzcd}$$

Property of π^i

- **Bounded**: $\|\pi^k u\|_{H(d^k)} \leq C\|u\|_{H(d^k)}$ or $\|\pi^k u\|_{L^2} \leq C\|u\|_{L^2}$ for any $u \in H\Lambda^k(\Omega)$(or $u \in L^2\Lambda^k(\Omega)$).
- **Commuting**: $\pi^{k+1} \circ d^k = d^k \circ \pi^k$.
- **Projection**: $\pi^k u_h = u_h$ for any $u_h \in \Lambda_h^k(\Omega)$.
- **Local**: For each element T, $\pi^i u|_T$ depend solely on a patch of T

Exsting interploation

- Non-local: [Schöberl, 2005], [Arnold & Falk & Winther, 2006], [Christiansen & Winther, 2008], etc.
- Local: [Falk & Winther, 2014, 2015], [Arnold & Guzmán, 2021], [Ern & Guzmán & Potu & Vohralík, 2025]

Other complexes

BGG complexes ([Arnold & Hu, 2021])

- **Hessian complex**

$$\begin{tikzcd} 0 \arrow[r] & H^2(\Omega) \arrow[r, "\mathrm{hess}"] & H(\mathrm{curl}, \Omega; \mathbb{S}) \arrow[r, "\mathrm{curl}"] & H(\mathrm{div}, \Omega; \mathbb{T}) \arrow[r, "\mathrm{div}"] & L^2(\Omega; \mathbb{R}^3) \arrow[r] & 0 \end{tikzcd}$$

- **divdiv complex**

$$\begin{tikzcd} 0 \arrow[r] & H^1(\Omega; \mathbb{R}^3) \arrow[r, "\mathrm{dev\,grad}"] & H(\mathrm{sym\,curl}, \Omega; \mathbb{T}) \arrow[r, "\mathrm{sym\,curl}"] & H(\mathrm{div\,div}, \Omega; \mathbb{S}) \arrow[r, "\mathrm{div\,div}"] & L^2(\Omega) \arrow[r] & 0 \end{tikzcd}$$

- **elasticity complex**

$$\begin{tikzcd} 0 \arrow[r] & H^1(\Omega; \mathbb{R}^3) \arrow[r, "\mathrm{sym\,grad}"] & H(\mathrm{inc}, \Omega; \mathbb{S}) \arrow[r, "\mathrm{inc}"] & H(\mathrm{div}, \Omega; \mathbb{S}) \arrow[r, "\mathrm{div}"] & L^2(\Omega; \mathbb{R}^3) \arrow[r] & 0 \end{tikzcd}$$

- **conformal deformation complex, conformal Hessian complex**

Extra smoothness

- **Stokes complex**

$$\begin{tikzcd} 0 \arrow[r] & H^2(\Omega) \arrow[r, "\mathrm{grad}"] & H^1(\mathrm{curl}, \Omega; \mathbb{R}^3) \arrow[r, "\mathrm{curl}"] & H^1(\Omega; \mathbb{R}^3) \arrow[r, "\mathrm{div}"] & L^2(\Omega) \arrow[r] & 0 \end{tikzcd}$$

Finite element complexes

Finite element discretizations

- Hessian complex: [Hu & Liang, 2021],etc.
- divdiv complex: [Chen & Huang, 2022divdiv], [Hu & Liang & Ma, 2022], [Hu & Liang & Ma & Zhang, 2024],etc.
- elasticity complex: [Christiansen & Hu & Hu, 2018], [Chen & Huang, 2022elasticity], [Christiansen & Gopalakrishnan & Guzmán & Hu, 2024],etc.
- Stokes complex: [Falk & Neilan, 2013], [Neilan, 2015], [Fu & Guzmán & Neilan, 2020], [Hu & Zhang & Zhang, 2022], [Guzmán & Lischke & Neilan, 2022],etc.
- conformal deformation complex: [Hu & Lin & Shi, 2023], [Huang, 2025]
- conformal Hessian complex: [Guo & Hu & Lin, 2025], [Huang, 2025]

Exactness: These finite element complexes are exact on a contractible domain.

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- Stokes complex: [Falk & Neilan, 2013], [Neilan, 2015], [Fu & Guzmán & Neilan, 2020], [Hu & Zhang & Zhang, 2022], [Guzmán & Lischke & Neilan, 2022],etc.
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Question I: What about the finite element cohomology on a non-contractible domain?

Question II: How can one construct local bounded commuting projections for nonstandard finite element complexes?

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Key ingredient I: Generalized de Rham map

For each smooth BGG complex (e.g., Hessian, divdiv, elasticity, conformal), assume its cohomology is isomorphic to $\mathcal{Z} \otimes \mathcal{H}_{dR}(\Omega)$

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(\Omega; \mathbb{X}^0) \arrow[r, "d^0"] & C^\infty(\Omega; \mathbb{X}^1) \arrow[r, "d^1"] & \cdots \arrow[r, "d^{n-1}"] & C^\infty(\Omega; \mathbb{X}^n) \arrow[r] & 0
\end{tikzcd}$$

Key ingredient I: Generalized de Rham map (generalized currents)

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(\Omega; \mathbb{X}^0) \arrow[r, "d^0"] \arrow[d, "\Upsilon_0"'] & C^\infty(\Omega; \mathbb{X}^1) \arrow[r, "d^1"] \arrow[d, "\Upsilon_1"'] & \cdots \arrow[r, "d^{n-1}"] \arrow[d, "\cdot"'] & C^\infty(\Omega; \mathbb{X}^n) \arrow[r] \arrow[d, "\Upsilon_n"'] & 0 \\
0 \arrow[r] & \mathcal{Z} \otimes \Delta_0' \arrow[r, "\delta"] & \mathcal{Z} \otimes \Delta_1' \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & \mathcal{Z} \otimes \Delta_n' \arrow[r] & 0
\end{tikzcd}$$

Properties of Υ_i:

(1) **Surjectivity**: $\Upsilon_i : C^\infty(\Omega; \mathbb{X}^i) \to \mathcal{Z} \otimes \Delta_i'$.

(2) **Commuting property**: $\delta \circ \Upsilon_i = \Upsilon_{i+1} \circ d^i$.

3D Hessian complex

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(\Omega) \arrow[r, "\mathrm{hess}"] \arrow[d, "\Upsilon_0"'] & C^\infty(\Omega;\mathbb{S}) \arrow[r, "\mathrm{curl}"] \arrow[d, "\Upsilon_1"'] & C^\infty(\Omega;\mathbb{T}) \arrow[r, "\mathrm{div}"] \arrow[d, "\Upsilon_2"'] & C^\infty(\Omega;\mathbb{R}^3) \arrow[r] \arrow[d, "\Upsilon_3"'] & 0 \\
0 \arrow[r] & \mathbb{R}^4\otimes\Delta_0' \arrow[r, "\delta"] & \mathbb{R}^4\otimes\Delta_1' \arrow[r, "\delta"] & \mathbb{R}^4\otimes\Delta_2' \arrow[r, "\delta"] & \mathbb{R}^4\otimes\Delta_3' \arrow[r] & 0
\end{tikzcd}$$

Let $\{\psi_i\}_{1\le i\le 4}$ be a basis of $\mathcal{RT}^3 = \{\boldsymbol{a} + b\boldsymbol{x} : \boldsymbol{a}\in\mathbb{R}^3, b\in\mathbb{R}\}$,

$$[\Upsilon_0[u](x)]_i = (\psi_i\cdot\nabla u)(x) - \tfrac{1}{3}(\operatorname{div}\psi_i\cdot u)(x), \quad x\in\Delta_0, 1\le i\le 4$$

$$[\Upsilon_1[\boldsymbol{\sigma}](e)]_i = \int_e (\boldsymbol{\sigma}\boldsymbol{t}_e)\cdot\psi_i, \quad e\in\Delta_1, 1\le i\le 4$$

$$[\Upsilon_2[\boldsymbol{\tau}](f)]_i = \int_f (\boldsymbol{\tau}\boldsymbol{n}_f)\cdot\psi_i, \quad f\in\Delta_2, 1\le i\le 4$$

$$[\Upsilon_3[\boldsymbol{q}](T)]_i = \int_T \boldsymbol{q}\cdot\psi_i, \quad T\in\Delta_3, 1\le i\le 4$$

3D divdiv complex

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(\Omega;\mathbb{R}^3) \arrow[r, "\operatorname{dev}\nabla"] \arrow[d, "\Upsilon_0"'] & C^\infty(\Omega;\mathbb{T}) \arrow[r, "\operatorname{sym}\operatorname{curl}"] \arrow[d, "\Upsilon_1"'] & C^\infty(\Omega;\mathbb{S}) \arrow[r, "\operatorname{divdiv}"] \arrow[d, "\Upsilon_2"'] & C^\infty(\Omega) \arrow[r] \arrow[d, "\Upsilon_3"'] & 0 \\
0 \arrow[r] & \mathbb{R}^4 \otimes \Delta_0' \arrow[r, "\delta"] & \mathbb{R}^4 \otimes \Delta_1' \arrow[r, "\delta"] & \mathbb{R}^4 \otimes \Delta_2' \arrow[r, "\delta"] & \mathbb{R}^4 \otimes \Delta_3' \arrow[r] & 0
\end{tikzcd}$$

Let $\{\psi_i\}_{1\le i\le 4}$ be a basis of $P_1(\Omega)$,

$$[\Upsilon_0[\boldsymbol{u}](x)]_i = \tfrac{1}{3}(\psi_i \cdot \operatorname{div}\boldsymbol{u})(x) - (\nabla\psi_i \cdot \boldsymbol{u})(x), \quad x \in \Delta_0, 1 \le i \le 4$$

$$[\Upsilon_1[\boldsymbol{\tau}](e)]_i = \int_e \tfrac{1}{2}(\operatorname{div}\boldsymbol{\tau}^T \cdot \boldsymbol{t}_e) \cdot \psi_i - (\boldsymbol{\tau}\boldsymbol{t}_e) \cdot \nabla\psi_i, \quad e \in \Delta_1, 1 \le i \le 4$$

$$[\Upsilon_2[\boldsymbol{\sigma}](f)]_i = \int_f (\operatorname{div}\boldsymbol{\sigma} \cdot \boldsymbol{n}_f) \cdot \psi_i - (\boldsymbol{\sigma}\boldsymbol{n}_f) \cdot \nabla\psi_i, \quad f \in \Delta_2, 1 \le i \le 4$$

$$[\Upsilon_3[q](T)]_i = \int_T q \cdot \psi_i, \quad T \in \Delta_3, 1 \le i \le 4$$

3D elasticity complex

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(\Omega;\mathbb{R}^3) \arrow[r, "\operatorname{sym}\nabla"] \arrow[d, "\Upsilon_0"'] & C^\infty(\Omega;\mathbb{S}) \arrow[r, "\operatorname{inc}"] \arrow[d, "\Upsilon_1"'] & C^\infty(\Omega;\mathbb{S}) \arrow[r, "\operatorname{div}"] \arrow[d, "\Upsilon_2"'] & C^\infty(\Omega;\mathbb{R}^3) \arrow[r] \arrow[d, "\Upsilon_3"'] & 0 \\
0 \arrow[r] & \mathbb{R}^6 \otimes \Delta_0' \arrow[r, "\delta"] & \mathbb{R}^6 \otimes \Delta_1' \arrow[r, "\delta"] & \mathbb{R}^6 \otimes \Delta_2' \arrow[r, "\delta"] & \mathbb{R}^6 \otimes \Delta_3' \arrow[r] & 0
\end{tikzcd}$$

Let $\{\psi_i\}_{1\le i\le 6}$ be a basis of $\mathcal{RM}^3 = \{\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x} : \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3\}$,

$$[\Upsilon_0[\boldsymbol{u}](x)]_i = \tfrac{1}{2}(\psi_i \cdot \operatorname{curl}\boldsymbol{u})(x) + \tfrac{1}{2}(\operatorname{curl}\psi_i \cdot \boldsymbol{u})(x), \quad x \in \Delta_0, 1 \le i \le 6$$

$$[\Upsilon_1[\boldsymbol{\sigma}](e)]_i = \int_e \tfrac{1}{2}(\boldsymbol{\sigma}\boldsymbol{t}_e) \cdot \operatorname{curl}\psi_i + ((\operatorname{curl}\boldsymbol{\sigma})\boldsymbol{t}_e) \cdot \psi_i, \quad e \in \Delta_1, 1 \le i \le 6$$

$$[\Upsilon_2[\boldsymbol{\tau}](f)]_i = \int_f (\boldsymbol{\tau}\boldsymbol{n}_f) \cdot \psi_i, \quad f \in \Delta_2, 1 \le i \le 6$$

$$[\Upsilon_3[\boldsymbol{q}](T)]_i = \int_T \boldsymbol{q} \cdot \psi_i, \quad T \in \Delta_3, 1 \le i \le 6$$

Finite element complex with trace structures

Given BGG complex (e.g., Hessian, divdiv, elasticity, conformal)

$$0 \longrightarrow C^\infty(\Omega;\mathbb{X}^0) \xrightarrow{d^0} C^\infty(\Omega;\mathbb{X}^1) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} C^\infty(\Omega;\mathbb{X}^n) \longrightarrow 0$$

- **Shape function space**: For any sub-simplex $\tau \in \Delta(\mathcal{T})$, given finite dimensional shape function spaces $A^k(\tau), 0 \leq k \leq n$
- **Trace structure**: For each $\eta \trianglelefteq \tau \trianglelefteq \sigma$, and $\sigma \in \Delta_n$, there exists linear trace operators for $0 \leq k \leq n$

  $$\operatorname{tr}^k_{\sigma\to\tau} : A^k(\sigma) \to A^k(\tau), \quad \operatorname{tr}^k_{\tau\to\eta} : A^k(\tau) \to A^k(\eta),$$

  such that

  $$\operatorname{tr}^k_{\sigma\to\eta} u = 0 \text{ implies } \operatorname{tr}^k_{\tau\to\eta} \circ \operatorname{tr}^k_{\sigma\to\tau} u = 0.$$

- **Localized generalized currents**: The k-th generalized currents are a part of the trace structures of k-th finite element, i.e., for any $\tau \in \Delta_k$, there exists localized generalized currents $\tilde{\Upsilon}_\tau : A^k(\tau) \to \mathcal{Z}$ such that

  $$\widetilde{\Upsilon}_\tau(\operatorname{tr}_{\sigma\to\tau} w) = \Upsilon_k[w](\tau) \text{ for any } w \in A^k(\sigma), \sigma \in \Delta_n, \text{ and } \tau \trianglelefteq \sigma.$$

- **Trace complexes**: For any simplex $\tau \in \Delta(\mathcal{T})$, there exists a differential operator $d_\tau^k : A^k(\tau) \to A^{k+1}(\tau)$ such that $d_\tau^{k+1} \circ d_\tau^k = 0$,

$$0 \longrightarrow A^0(\tau) \xrightarrow{d_\tau^0} A^1(\tau) \xrightarrow{d_\tau^1} \cdots \xrightarrow{d_\tau^{n-1}} A^n(\tau) \longrightarrow 0.$$

For any $\tau \trianglelefteq \sigma$ with $\sigma \in \Delta_n$, it holds $d_\sigma^k = d^k|_\sigma$ and

$$d_\tau^k \operatorname{tr}_{\sigma\to\tau}^k u = \operatorname{tr}_{\sigma\to\tau}^{k+1} d_\sigma^k u, \quad \forall u \in A^k(\sigma), 0 \le k \le n-1.$$

$$\begin{tikzcd}
\sigma \in \Delta_n & \cdots \arrow[r] & A^{k-1}(\sigma) \arrow[r, "d_\sigma^{k-1}"] \arrow[d, "\operatorname{tr}_{\sigma\to\tau}^{k-1}"] \arrow[dd, bend right, "\operatorname{tr}_{\sigma\to\eta}^{k-1}"'] & A^k(\sigma) \arrow[r, "d_\sigma^k"] \arrow[d, "\operatorname{tr}_{\sigma\to\tau}^k"] \arrow[dd, bend right, "\operatorname{tr}_{\sigma\to\eta}^k"'] & A^{k+1}(\sigma) \arrow[r] \arrow[d, "\operatorname{tr}_{\sigma\to\tau}^{k+1}"] \arrow[dd, bend left, "\operatorname{tr}_{\sigma\to\eta}^{k+1}"] & \cdots \\
\tau \trianglelefteq \sigma & \cdots \arrow[r] & A^{k-1}(\tau) \arrow[r, "d_\tau^{k-1}"] \arrow[d, "\operatorname{tr}_{\tau\to\eta}^{k-1}"] & A^k(\tau) \arrow[r, "d_\tau^k"] \arrow[d, "\operatorname{tr}_{\tau\to\eta}^k"] & A^{k+1}(\tau) \arrow[r] \arrow[d, "\operatorname{tr}_{\tau\to\eta}^{k+1}"] & \cdots \\
\eta \trianglelefteq \tau & \cdots \arrow[r] & A^{k-1}(\eta) \arrow[r, "d_\eta^{k-1}"] & A^k(\eta) \arrow[r, "d_\eta^k"] & A^{k+1}(\eta) \arrow[r] & \cdots
\end{tikzcd}$$

- **Global space**: Define the global space

$$\boldsymbol{A}^k = \{u \in L^2(\Omega;\mathbb{X}^k) : u|_\sigma \in A^k(\sigma), \operatorname{tr}^k_{\sigma\to\tau} u|_\sigma = \operatorname{tr}^k_{\sigma'\to\tau} u|_{\sigma'},\\ \text{for all } \tau \trianglelefteq \sigma,\sigma' \text{ and } \sigma,\sigma' \in \Delta_n\}.$$

Finite element complex:

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{A} \arrow[r, "d^0"] & \boldsymbol{A}^1 \arrow[r, "d^1"] & \cdots \arrow[r, "d^{n-1}"] & \boldsymbol{A}^n \arrow[r] & 0. \end{tikzcd}$$

- **Local bubble space**: Define the bubble space $B^k(\tau) \subseteq \operatorname{tr}^k_{\sigma\to\tau} A^k(\sigma)$ with $\sigma \in \Delta_n$, such that for any $a \in B^k(\tau)$:
  (1) **Vanishing traces**: $\operatorname{tr}^k_{\tau\to\eta} a = 0$ for all $\eta \trianglelefteq \tau$ and $\eta \neq \tau$.
  (2) **Vanishing current**: $\widetilde{\Upsilon}_\tau(a) = 0$ if $\dim\tau = k$.

  Assume that the following sequence of bubble spaces

$$\begin{tikzcd} 0 \arrow[r] & B^0(\tau) \arrow[r, "d^0_\tau"] & B^1(\tau) \arrow[r, "d^1_\tau"] & \cdots \arrow[r, "d^{n-1}_\tau"] & B^n(\tau) \arrow[r] & 0 \end{tikzcd}$$

  forms an exact complex for any $\tau \in \Delta(\mathcal{T})$.

DOFs and Basis function: for any $w \in \boldsymbol{A}^k, 0 \leq k \leq n$

- **Skeletal Component**:
  - **DOFs**: $\Upsilon_k[w](\tau)$ for all $\tau \in \Delta_k$.
  - **Basis space $\boldsymbol{S}^k(\mathcal{T})$**: span{basis functions dual to skeletal DOFs}.

- **Bubble Component**:
  - **DOFs**: $\langle \operatorname{tr}_{\sigma \to \tau} w, \phi \rangle_{B^k(\tau)}$ for all $\tau \in \Delta(\mathcal{T})$ and $\phi \in B^k(\tau)$.
  - **Basis space** $\mathbb{B}^k(\tau)$: span{basis functions dual to bubble DOFs}.

Direct sum decomposition of the global finite element space:

$$\boldsymbol{A}^k = \boldsymbol{S}^k(\mathcal{T}) \oplus \bigoplus_{\tau \in \Delta(\mathcal{T})} \mathbb{B}^k(\tau).$$

Key ingredient II: Geometric decomposition

Key ingredient II: Geometric decomposition of finite element complex

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{A}^0 \arrow[r, "d^0"] & \boldsymbol{A}^1 \arrow[r, "d^1"] & \cdots \arrow[r, "d^{n-1}"] & \boldsymbol{A}^n \arrow[r] & 0 \end{tikzcd}$$

is a direct sum of the Skeletal complex

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{S}^0(\mathcal{T}) \arrow[r, "d^0"] & \boldsymbol{S}^1(\mathcal{T}) \arrow[r, "d^1"] & \cdots \arrow[r, "d^{n-1}"] & \boldsymbol{S}^n(\mathcal{T}) \arrow[r] & 0 \end{tikzcd}$$

and the Bubble complex

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{B}^0(\tau) \arrow[r, "d^0"] & \mathbb{B}^1(\tau) \arrow[r, "d^1"] & \cdots \arrow[r, "d^{n-1}"] & \mathbb{B}^n(\tau) \arrow[r] & 0 \end{tikzcd}$$

with all $\tau \in \Delta(\mathcal{T})$.

Cohomology of finite element complex

Cohomology of subcomplex

- **Skeletal complex**

$$\begin{tikzcd}
0 \arrow[r] & S^0(\mathcal{T}) \arrow[r, "d^0"] \arrow[d, swap, "\Upsilon_0"] & S^1(\mathcal{T}) \arrow[r, "d^1"] \arrow[d, swap, "\Upsilon_1"] & \cdots \arrow[r, "d^{n-1}"] \arrow[d, swap, "\cdot"] & S^n(\mathcal{T}) \arrow[r] \arrow[d, swap, "\Upsilon_3"] & 0 \\
0 \arrow[r] & \mathcal{Z} \otimes \Delta_0' \arrow[r, "\delta"] & \mathcal{Z} \otimes \Delta_1' \arrow[r, "\delta"] & \cdots \arrow[r, "\delta"] & \mathcal{Z} \otimes \Delta_n' \arrow[r] & 0
\end{tikzcd}$$

  here $S^k(\mathcal{T}) \overset{\Upsilon_k}{\cong} \mathcal{Z} \otimes \Delta_k', 0 \leq k \leq n$.

- **Bubble complex**

$$0 \longrightarrow \mathbb{B}^0(\tau) \xrightarrow{d^0} \mathbb{B}^1(\tau) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathbb{B}^n(\tau) \longrightarrow 0$$

  is local and exact.

Conclusion I: The cohomology of the finite element complex

$$0 \longrightarrow \boldsymbol{A}^0 \xrightarrow{d^0} \boldsymbol{A}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \boldsymbol{A}^n \longrightarrow 0$$

is isomorphic to $\mathcal{Z} \otimes \mathcal{H}_{dR}(\Omega)$.

Commuting interpolations for finite element complex

Conclusion II: Construction of local bounded commuting projections $\pi^k : L^2(\Omega; \mathbb{X}^k) \to \boldsymbol{A}^k$:

$$\pi^k(w) = \pi^k_{\boldsymbol{S}}(w) + \sum_{\tau \in \Delta(\mathcal{T})} \pi^k_{\boldsymbol{B},\tau}(w),$$

here

$$\pi^k_{\boldsymbol{S}}(w) = \sum_{\tau \in \Delta_k} \sum_{i=1}^{\dim \mathcal{Z}} (w, \Xi^i_\tau)_{\mathrm{star}^1(\tau)} \mathsf{E}^k_{\mathsf{S}}(z_i \otimes \tau) \in S^k(\mathcal{T}),$$

and

$$\pi^k_{\boldsymbol{B},\tau}(w) = \sum_{i=1}^{\mathcal{N}_{\tau,k}} (w, d^*\Omega^i_{\tau,k+1})_{\mathrm{star}(\tau)} \mathsf{E}^k_\tau(\phi^i_{\tau,k}) + \sum_{i=1}^{\mathcal{N}_{\tau,k-1}} (w, \Omega^i_{\tau,k})_{\mathrm{star}(\tau)} \mathsf{E}^k_\tau(d\phi^i_{\tau,k-1}) \in \mathbb{B}^k(\tau).$$

with some $\Xi^i_\tau \in H^M_0(\mathrm{star}^1(\tau)) \otimes \mathbb{X}^{\dim \tau}$ and $\Omega^i_{\tau,k} \in C^{2M}_0(\mathrm{star}(\tau)) \otimes \mathbb{X}^k$.

Ref:[Arnold & Guzmán, 2021], [Ern & Guzmán & Potu & Vohralík, 2025].

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2D Stokes complex

Consider the following finite element 2D stokes complex ([Falk & Neilan, 2013])

$$0 \longrightarrow \mathbf{Ar}_k(\Omega) \xrightarrow{\text{curl}} \mathbf{Hm}_{k-1}(\Omega) \xrightarrow{\text{div}} \mathcal{P}^0_{k-2}(\mathcal{T}_h) \longrightarrow 0$$

with $k \geq 5$. Here

$$\mathbf{Ar}_k(\Omega) = \{u \in H^2(\Omega) : u|_f \in P_k(f), u \text{ is } C^2 \text{ at each vertex } x \in \Delta_0\}$$

$$\mathbf{Hm}_{k-1}(\Omega) = \{\boldsymbol{v} \in H^1(\Omega; \mathbb{R}^2) : \boldsymbol{v}|_f \in P_{k-1}(f, \mathbb{R}^2), \boldsymbol{v} \text{ is } C^1 \text{ at each vertex } x \in \Delta_0$$

$$\mathcal{P}^0_{k-2}(\mathcal{T}_h) = \{p \in L^2(\Omega) : p|_f \in P_{k-2}(f), p \text{ is } C^0 \text{ at each vertex } x \in \Delta_0\}$$

Recall the generalized currents (de Rham map)

$$\Upsilon_x : u \mapsto u(x), \quad \Upsilon_e : \boldsymbol{v} \mapsto \int_e \boldsymbol{v} \cdot \boldsymbol{n}_e, \quad \Upsilon_f : p \mapsto \int_f p$$

Another set of degrees of freedom

	$\mathbf{Ar}_k(\Omega)$	$\xrightarrow{\text{curl}}$	$\mathbf{Hm}_{k-1}(\Omega)$	$\xrightarrow{\text{div}}$	$\mathcal{P}^0_{k-2}(\mathcal{T}_h)$
Skeletal	$\Upsilon_x(u)$		$\Upsilon_e(v)$		$\Upsilon_f(p)$
Vertex	$\nabla u(x), \nabla^2 u(x)$		$\nabla \boldsymbol{v}(x), \boldsymbol{v}(x)$		$p(x)$
Edge(1)	$(\frac{\partial u}{\partial t}, \frac{\partial b_e^1}{\partial t})_e$		$(\boldsymbol{v}\cdot\boldsymbol{n}, b_e^2)_e$		
Edge(2)	$(\frac{\partial u}{\partial n}, b_e^{1,*})_e$		$(v\cdot\boldsymbol{t}, b_e^{2,*})_e$		
Face	$(\operatorname{curl} u, \operatorname{curl} b_f^1)_f$		$\langle\langle \boldsymbol{v}, \boldsymbol{b}_f^2\rangle\rangle_f$		$(p, b_f^3)_f$

Here

$$b_e^1 \in (\lambda_0\lambda_1)^3 P_{k-6}(e),$$

$$b_e^2 \in (\lambda_0\lambda_1)^2 P_{k-5}(e)/\mathbb{R},$$

$$b_e^{1,*} \in (\lambda_0\lambda_1)^2 P_{k-5}(e),$$

$$b_e^{2,*} \in (\lambda_0\lambda_1)^2 P_{k-5}(e),$$

$$b_f^1 \in B^2_{f,k-6} := (\lambda_0\lambda_1\lambda_2)^2 P_{k-6}(f),$$

$$\boldsymbol{b}_f^2 \in (\lambda_0\lambda_1\lambda_2) P_{k-4}(f;\mathbb{R}^2),$$

$$b_f^3 \in \{p \in P_{k-2}(f) : p \text{ vanishes at vertices of } f\}/\mathbb{R},$$

and $\langle\langle \boldsymbol{u}, \boldsymbol{v}\rangle\rangle_f = (P_{\mathrm{im}\ \mathrm{curl} B^2_{f,k-6}}\boldsymbol{u}, P_{\mathrm{im}\ \mathrm{curl} B^2_{f,k-6}}\boldsymbol{v})_f + (\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v})_f$

Separation of DOFs

- For each $u \in \mathbf{Ar}_k(\Omega)$, $\operatorname{curl} u \in \mathbf{Hm}_{k-1}(\Omega)$ and

  "**Skeletal**" DOFs of u vanish $\Rightarrow$ "**Skeletal**" DOFs of $\operatorname{curl} u$ vanish

- For each vertex $x \in \Delta_0$

  "**Vertex**" DOFs of u at x vanish $\Rightarrow$ "**Vertex**" DOFs of $\operatorname{curl} u$ at x vanish

- For each edge $e \in \Delta_1$

  "**Edge**" DOFs of u on e vanish $\Rightarrow$ "**Edge**" DOFs of $\operatorname{curl} u$ on e vanish

- For each face $f \in \Delta_2$

  "**Face**" DOFs of u on f vanish $\Rightarrow$ "**Face**" DOFs of $\operatorname{curl} u$ on f vanish

This also holds for each $\boldsymbol{v} \in \mathbf{Hm}_{k-1}(\Omega)$ and $\operatorname{div} \boldsymbol{v} \in \mathcal{P}^0_{k-2}(\mathcal{T}_h)$.

Geometric decomposition of complex

Geometric decomposition of finite elements

$$\mathbf{Ar}_k(\Omega) := A^S \oplus (\oplus_{x\in\Delta_0} A^x) \oplus (\oplus_{e\in\Delta_1} A^e) \oplus (\oplus_{f\in\Delta_2} A^f)$$

$$\mathbf{Hm}_{k-1}(\Omega) := H^S \oplus (\oplus_{x\in\Delta_0} H^x) \oplus (\oplus_{e\in\Delta_1} H^e) \oplus (\oplus_{f\in\Delta_2} H^f)$$

$$\mathcal{P}^0_{k-2}(\mathcal{T}_h) := P^S \oplus (\oplus_{x\in\Delta_0} P^x) \oplus (\oplus_{e\in\Delta_1} P^e) \oplus (\oplus_{f\in\Delta_2} P^f)$$

Geometric decomposition of 2D Stokes complex: The 2D Stokes complex

$$0 \longrightarrow \mathbf{Ar}_k(\Omega) \xrightarrow{\text{curl}} \mathbf{Hm}_{k-1}(\Omega) \xrightarrow{\text{div}} \mathcal{P}^0_{k-2}(\mathcal{T}_h) \longrightarrow 0$$

is a direct sum of the Skeletal complex

$$0 \longrightarrow A^S \xrightarrow{\text{curl}} H^s \xrightarrow{\text{div}} P^S \longrightarrow 0$$

and Bubble complexes

$$0 \longrightarrow A^\tau \xrightarrow{\text{curl}} H^\tau \xrightarrow{\text{div}} P^\tau \longrightarrow 0$$

with each $\tau \in \Delta(\mathcal{T}_h)$.

Cohomology of 2D Stokes complex

- For each $\tau \in \Delta(\mathcal{T}_h)$, the Bubble complex

$$\begin{tikzcd} 0 \arrow[r] & A^\tau \arrow[r, "\mathrm{curl}"] & H^\tau \arrow[r, "\mathrm{div}"] & P^\tau \arrow[r] & 0 \end{tikzcd}$$

  is local and exact.

- The Skeletal complex

$$\begin{tikzcd} 0 \arrow[r] & A^S \arrow[r, "\mathrm{curl}"] & H^s \arrow[r, "\mathrm{div}"] & P^S \arrow[r] & 0 \end{tikzcd}$$

  is isomorphic to the simplicial cochain complex

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \otimes \Delta_0' \arrow[r, "\delta"] & \mathbb{R} \otimes \Delta_1' \arrow[r, "\delta"] & \mathbb{R} \otimes \Delta_2' \arrow[r] & 0 \end{tikzcd}$$

  the isomorphism is deduced by $\{\Upsilon_\bullet\}$.

- **Conclusion**: The cohomology of the 2D finite element Stokes complex is isomorphic to $\mathcal{H}_{dR}(\Omega)$.

3D Hessian complex

Consider the following finite element 3D Hessian complex ([Hu & Liang, 2021])

$$\begin{tikzcd} 0 \arrow[r] & U_h \arrow[r, "\mathrm{hess}"] & \Sigma_h \arrow[r, "\mathrm{curl}"] & V_h \arrow[r, "\mathrm{div}"] & Q_h \arrow[r] & 0 \end{tikzcd}$$

For $k \geq 9$, here

$$\begin{aligned} U_h = \{u \in H^2(\Omega) : u|_T \in P_k(T), u \text{ is } C^4 \text{ at each vertex } x \in \Delta_0, \\ C^2 \text{ on each edge } e \in \Delta_1\} \end{aligned}$$

$$\begin{aligned} \Sigma_h = \{\boldsymbol{\sigma} \in H(\mathrm{curl}, \Omega; \mathbb{S}) : \boldsymbol{\sigma}|_T \in P_{k-2}(T; \mathbb{S}), \boldsymbol{\sigma} \text{ is } C^2 \text{ at each vertex } x \in \Delta_0, \\ C^0 \text{ on each edge } e \in \Delta_1\} \end{aligned}$$

$$V_h = \{\boldsymbol{v} \in H(\mathrm{div}, \Omega; \mathbb{T}) : \boldsymbol{v}|_T \in P_{k-3}(T; \mathbb{T}), \boldsymbol{\sigma} \text{ is } C^1 \text{ at each vertex } x \in \Delta_0\}$$

$$Q_h = \{\boldsymbol{q} \in L^2(\Omega; \mathbb{R}^3) : \boldsymbol{q}|_T \in P_{k-4}(T; \mathbb{R}^3), \boldsymbol{\sigma} \text{ is } C^0 \text{ at each vertex } x \in \Delta_0\}$$

Another degrees of freedom

	U_h	$\xrightarrow{\text{hess}}$	Σ_h	$\xrightarrow{\text{curl}}$	V_h	$\xrightarrow{\text{div}}$	Q_h
Skeletal	$\Upsilon_x(u)$		$\Upsilon_e(\boldsymbol{\sigma})$		$\Upsilon_f(\boldsymbol{v})$		$\Upsilon_T(\boldsymbol{q})$
Vertex	$D^\alpha u(x), 2 \le \|\alpha\| \le 4$		$D^\beta \boldsymbol{\sigma}(x), \|\beta\| \le 2$		$\boldsymbol{v}(\boldsymbol{x}), \nabla \boldsymbol{v}(x)$		$\boldsymbol{q}(x)$
Edge(1)	$(\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 b_e^1}{\partial t^2})_e$		$(\boldsymbol{t}^T \boldsymbol{\sigma} \boldsymbol{t}, b_{\boldsymbol{e}}^2)_e$				
Edge(2)	$(\frac{\partial^2 u}{\partial n \partial t}, \frac{\partial b_e^{1,*}}{\partial t})_e$		$(\boldsymbol{n}_\pm^T \boldsymbol{\sigma} \boldsymbol{t}, b_e^{2,*})_e$				
Edge(3)	$(\frac{\partial^2 u}{\partial n_\pm \partial n_\pm}, \frac{\partial b_e^{1,**}}{\partial t})_e$		$(\boldsymbol{n}_\pm^T \boldsymbol{\sigma} \boldsymbol{n}_\pm, b_e^{2,**})_e$				
Face(1)	$(\nabla_f^2 u, \nabla_f^2 b_f^1)_f$		$\langle\langle \boldsymbol{\sigma}, \boldsymbol{b}_f^2 \rangle\rangle_f^1$		$(E_f(\boldsymbol{v}\boldsymbol{n}), b_f^3)_f$		
Face(2)	$(\nabla_f(\frac{\partial u}{\partial \boldsymbol{n}}), \nabla_f b_f^{1,*})_f$		$\langle\langle \boldsymbol{\sigma}, \boldsymbol{b}_f^{2,*} \rangle\rangle_f^2$		$(\boldsymbol{n}^T \boldsymbol{v} \boldsymbol{n}, b_f^{3,*})_f$		
Tetrahedron	$(\nabla^2 u, \nabla^2 b_T^1)_T$		$\langle\langle \boldsymbol{\sigma}, \boldsymbol{b}_T^2 \rangle\rangle_T^1$		$\langle\langle \boldsymbol{v}, \boldsymbol{b}_T^3 \rangle\rangle_T^2$		$(\boldsymbol{q}, \boldsymbol{b}_T^4)_T$

Redefine the generalized currents of 3D Hessian complex: let $\{\psi_i\}_{1 \le i \le 4}$ be a basis of $\mathcal{RT}^3 = \{\boldsymbol{a} + b\boldsymbol{x} : \boldsymbol{a} \in \mathbb{R}^3, b \in \mathbb{R}\}$

$$\Upsilon_x : U_h \to \mathbb{R}^4, \quad [\Upsilon_x(u)]_i = (\psi_i \cdot \nabla u)(x) - \tfrac{1}{3}(\operatorname{div} \psi_i \cdot u)(x),$$

$$\Upsilon_e : \Sigma_h \to \mathbb{R}^4, \quad [\Upsilon_e(\boldsymbol{\sigma})]_i = \int_e (\boldsymbol{\sigma} \boldsymbol{t}) \cdot \psi_i,$$

$$\Upsilon_f : V_h \to \mathbb{R}^4, \quad [\Upsilon_f(\boldsymbol{\tau})]_i = \int_f (\boldsymbol{\tau} \boldsymbol{n}) \cdot \psi_i,$$

$$\Upsilon_T : Q_h \to \mathbb{R}^4, \quad [\Upsilon_T(\boldsymbol{q})]_i = \int_T \boldsymbol{q} \cdot \psi_i.$$

Separation of DOFs

- For each $u \in U_h$, $\operatorname{hess} u \in \Sigma_h$ and

$$\text{“}\textbf{Skeletal}\text{” DOFs of } u \text{ vanish } \Rightarrow \text{“}\textbf{Skeletal}\text{” DOFs of } \operatorname{hess} u \text{ vanish}$$

- For each vertex $x \in \Delta_0$

$$\text{“}\textbf{Vertex}\text{” DOFs of } u \text{ at } x \text{ vanish } \Rightarrow \text{“}\textbf{Vertex}\text{” DOFs of } \operatorname{hess} u \text{ at } x \text{ vanish}$$

- For each edge $e \in \Delta_1$

$$\text{“}\textbf{Edge}\text{” DOFs of } u \text{ on } e \text{ vanish } \Rightarrow \text{“}\textbf{Edge}\text{” DOFs of } \operatorname{hess} u \text{ on } e \text{ vanish}$$

- For each face $f \in \Delta_2$

$$\text{“}\textbf{Face}\text{” DOFs of } u \text{ on } f \text{ vanish } \Rightarrow \text{“}\textbf{Face}\text{” DOFs of } \operatorname{hess} u \text{ on } f \text{ vanish}$$

- For each face $T \in \Delta_3$

$$\text{“}\textbf{Tetrahedron}\text{” DOFs of } u \text{ on } T \text{ vanish } \Rightarrow \text{“}\textbf{Tetrahedron}\text{” DOFs of } \operatorname{hess} u \text{ on } T \text{ vanish}$$

This also holds for each $\boldsymbol{\sigma} \in \Sigma_h$ and $\boldsymbol{v} \in V_h$.

Geometric decomposition of complex

Geometric decomposition of finite elements

$$
\begin{aligned}
U_h &:= U^S \oplus (\oplus_{x\in\Delta_0} U^x) \oplus (\oplus_{e\in\Delta_1} U^e) \oplus (\oplus_{f\in\Delta_2} U^f) \oplus (\oplus_{T\in\Delta_3} U^T)\\
\Sigma_h &:= \Sigma^S \oplus (\oplus_{x\in\Delta_0} \Sigma^x) \oplus (\oplus_{e\in\Delta_1} \Sigma^e) \oplus (\oplus_{f\in\Delta_2} \Sigma^f) \oplus (\oplus_{T\in\Delta_3} \Sigma^T)\\
V_h &:= V^S \oplus (\oplus_{x\in\Delta_0} V^x) \oplus (\oplus_{e\in\Delta_1} V^e) \oplus (\oplus_{f\in\Delta_2} V^f) \oplus (\oplus_{T\in\Delta_3} V^T)\\
Q_h &:= Q^S \oplus (\oplus_{x\in\Delta_0} Q^x) \oplus (\oplus_{e\in\Delta_1} Q^e) \oplus (\oplus_{f\in\Delta_2} Q^f) \oplus (\oplus_{T\in\Delta_3} Q^T)
\end{aligned}
$$

Geometric decomposition of 3D Hessian complex: The 3D Hessian complex

$$0 \longrightarrow U_h \xrightarrow{\text{hess}} \Sigma_h \xrightarrow{\text{curl}} V_h \xrightarrow{\text{div}} Q_h \longrightarrow 0$$

is a direct sum of the Skeletal complex

$$0 \longrightarrow U^S \xrightarrow{\text{hess}} \Sigma^S \xrightarrow{\text{curl}} V^S \xrightarrow{\text{div}} Q^S \longrightarrow 0$$

and Bubble complexes

$$0 \longrightarrow U^\tau \xrightarrow{\text{hess}} \Sigma^\tau \xrightarrow{\text{curl}} V^\tau \xrightarrow{\text{div}} Q^\tau \longrightarrow 0$$

with each $\tau \in \Delta(\mathcal{T}_h)$.

Cohomology of 3D Hessian complex

- For each $\tau \in \Delta(\mathcal{T}_h)$, the Bubble complex

$$\begin{tikzcd} 0 \arrow[r] & U^{\tau} \arrow[r, "\mathrm{hess}"] & \Sigma^{\tau} \arrow[r, "\mathrm{curl}"] & V^{\tau} \arrow[r, "\mathrm{div}"] & Q^{\tau} \arrow[r] & 0 \end{tikzcd}$$

  is local and exact.
- The Skeletal complex

$$\begin{tikzcd} 0 \arrow[r] & U^{S} \arrow[r, "\mathrm{hess}"] & \Sigma^{S} \arrow[r, "\mathrm{curl}"] & V^{S} \arrow[r, "\mathrm{div}"] & Q^{S} \arrow[r] & 0 \end{tikzcd}$$

  is isomorphic to the simplicial cochain complex

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}^4 \otimes \Delta_0' \arrow[r, "\delta"] & \mathbb{R}^4 \otimes \Delta_1' \arrow[r, "\delta"] & \mathbb{R}^4 \otimes \Delta_2' \arrow[r, "\delta"] & \mathbb{R}^4 \otimes \Delta_3' \arrow[r] & 0 \end{tikzcd}$$

  the isomorphism is deduced by $\{\Upsilon_\bullet\}$.
- **Conclusion**: The cohomology of the 3D finite element Hessian complex is isomorphic to $\mathbb{R}^4 \otimes \mathcal{H}_{dR}(\Omega)$.

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Conclusion and Extension

Conclusion: We provide a unified framework to investigate the cohomology and bounded interpolation of nonstandard finite element complexes, e.g. Stokes, Hessian, elasticity, divdiv.

Possible extension:

- Boundary treatment. ([Brubeck & Liang & Parker, in preparation])
- p-version, spline function, distributional function.
- Other BGG complexes whose cohomology is not isomorphic to $\mathcal{Z} \otimes \mathcal{H}_{dR}(\Omega)$, e.g., the curl-div and grad-curl complex.
- Other meshes.

Reference:

- J. Hu, Y. Liang, and T. Lin, Finite Element Complexes with Traces Structures: A unified framework for cohomology and bounded interpolation.[arXiv:2509.23788]

Thanks!